Total Roman Dominating in an Interval Graph with Alternate Cliques of Size 3

M. Reddappa  
Research Scholar  
Dept. of Mathematics, S.V.University, Tirupati-517502  

C. Jaya Subba Reddy  
Asst. Professor, Dept. of Mathematics, S.V.University, Tirupati-517502  

B. Maheswari  
Professor(Rtd.), Dept. of Applied Mathematics, S.P.M.V.Visvavidyalayam, Tirupati-517502

ABSTRACT
The theory of Graphs is an important branch of Mathematics that was developed exponentially. The theory of domination in graphs is rapidly growing area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

Interval graphs have drawn the attention of many researchers for over 40 years. They form a special class of graphs with many interesting properties and revealed their practical relevance for modeling problems arising in the real world. The theory of domination in graphs introduced by Ore [12] and Berge [4] has been ever green of graph theory today. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et.al. [1, 2].

In this paper a study of total domination and total Roman domination number of an interval graph with alternate cliques of size 3 is carried out.

Keywords  
Total domination number, Total Roman dominating function, Total Roman domination number, Total Roman dominating function, Interval family, Interval graph.

1. INTRODUCTION
Domination in graphs has been studied extensively in recent years and it is an important branch of Graph Theory. Allan, R.B. and Laskar, R.C.[3], Cockayne, E.J.and Hedetniemi, S.T [5] and many others have studied various domination parameters of graphs.

Let \( G(V,E) \) be a graph. A total dominating set of a graph \( G \) with no isolated vertices is a set \( S \) of vertices of \( G \) such that every vertex in \( V(G) \) is adjacent to at least one vertex in \( S \).

The minimum cardinality of a total dominating set is called as total domination number and it is denoted by \( \gamma_t(G) \). A total dominating set of \( G \) of cardinality \( \gamma_t(G) \) is called a \( \gamma_t(G) \)-set.

Total domination in graphs was introduced by Cockayne et.al. [7]. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book by Henning et al. [9].

We consider finite graphs without loops and multiple edges.

2. TOTAL ROMAN DOMINATING FUNCTION
The Roman dominating function of a graph \( G \) was defined by Cockayne et.al. [6]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart [10] entitled “Defend The Roman Empire!” and suggested even earlier by ReVelle [13].

A Roman dominating function on a graph \( G(V,E) \) is a function \( f: V \rightarrow \{0,1,2\} \) satisfying the condition that every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of a Roman dominating function is the value \( f(V) = \sum_{v \in V} f(v) \). The minimum weight of a Roman dominating function on a graph \( G \) is called as the Roman domination number of \( G \). It is denoted by \( \gamma_R(G) \).

A Roman dominating function \( f \) of \( G \) is called a Roman dominating function if the set \( V \) becomes a Roman dominating set, i.e., for each \( v \in V \) we have \( f(v) \geq 1 \).

A Roman dominating function on a graph is said to intersect each other if either \( V_1 \cap V_2 = \emptyset \) or \( V_1 \cap V_2 \neq \emptyset \) (with the additional property that the sub graph of \( G \) induced by \( V_1 \cup V_2 \) of positive weight under \( f \) has no isolated vertices.

The minimum weight of a total Roman dominating function is called as the total Roman domination number of \( G \) and it is denoted by \( \gamma_{tr}(G) \). A total Roman dominating function with minimum weight \( \gamma_{tr}(G) \) is called a \( \gamma_{tr}(G) \)-function. If \( \gamma_{tr}(G) = 2 \gamma_R(G) \) then \( G \) is called a total Roman graph.

3. INTERVAL GRAPH
Let \( I = \{i_1, i_2, i_3, \ldots, i_n\} \) be an interval family, where each \( i_l \) is an interval on the real line and \( i_l = [a_l, b_l] \) for \( i = 1,2,3,\ldots,n \). Here \( a_l \) is called the left end point and \( b_l \) is called the right end point of \( I \). Without loss of generality, we assume that all end points of the intervals in \( I \) are distinct numbers between 1 and 2n. Two intervals \( i = [a_i, b_i] \) and \( j = [a_j, b_j] \) are said to intersect each other if either \( a_i < b_i \) or \( a_j < b_j \). The intervals are labelled in the increasing order of their right end points. Let \( G(V,E) \) be a graph. \( G \) is called an interval graph if there is a 1-1 correspondence between \( V \) and \( I \) such that two vertices of \( G \) are joined by an edge in \( E \) if and
only if their corresponding intervals in $I$ intersect. If $i$ is an interval in $I$ the corresponding vertex in $\mathcal{G}$ is denoted by $v_i$. Consider the following interval family.

The corresponding interval graph is given by

Consider the following interval family. The corresponding interval graph is given by

Consider the following interval family.

The corresponding interval graph is given by

In what follows we consider interval graphs of this type. We observe that when $n = 3k + 3$ then the interval graph has adjacent cliques of size 3, $k = 1, 2, 3$ and when $n = 3k + 2$ then the interval graph has adjacent cliques of size 3 and the last clique has two adjacent edges and when $n = 3k + 4$ then the interval graph has adjacent cliques of size 3 and the last clique is adjacent with one edge, $k = 1, 2, 3$ and so on. We denote this type of interval graph by $\mathcal{G}$. The signed Roman domination is studied in the following for the interval graph $\mathcal{G}$. 

4. RESULTS

Theorem 4.1: Let $G$ be the Interval graph of with $n$ vertices and no isolated vertices, where $n \geq 7$. Then the total domination number of $G$ is

$$\gamma_t(G) = 2k + 1 \text{ for } n = 6k + 1$$

$$= 2k + 2 \text{ for } n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$$

where $k = 1, 2, 3, \ldots$ respectively.

Proof: Let $G$ be the Interval graph with vertex set \{v_1, v_2, v_3, v_4, \ldots, v_n\} and no isolated vertices, where $n \geq 7$.

Suppose $k = 1$. Then $n = 7$. We can easily see that $TD = \{v_2, v_4, v_6\}$ is a total dominating set of $G$. Now for

$n = 8, 9$ we see that $TD = \{v_2, v_4, v_6, v_7\}$ and for $n = 10, \ldots, 11$, $TD = \{v_2, v_4, v_6, v_7, v_{10}\}$ and for

$n = 11, 12, TD = \{v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$ are total dominating sets of $G$ respectively. Further we can show that all these sets are minimum total dominating sets. Therefore the total domination numbers of $G$ are $\gamma_t(G) = 3$ for $n = 7$ and

$\gamma_t(G) = 4$ for $n = 8, 9, 10, 11, 12$.

If $k = 2$, then $n = 13, 14, 15, 16, 17, 18$. Form is $13, TD = \{v_2, v_4, v_6, v_{10}, v_{12}\}$ and for $n = 14, 15, TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{13}\}$ and for $n = 16, TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{13}, v_{15}\}$ and for $n = 17, TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{13}, v_{15}, v_{16}\}$ are minimum total dominating sets of $G$. So the total domination numbers are $\gamma_t(G) = 5$ for $n = 13$ and

$\gamma_t(G) = 6$ for $n = 14, 15, 16, 17, 18$, respectively.

Similarly for $k = 3$ we have $n = 19, 20, 21, 22, 23, 24$. Then the minimum total dominating set of $G$ are

$TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{14}, v_{18}\}$ for $n = 19$;

$TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{14}, v_{16}, v_{18}\}$ for $n = 20, 21$;

$TD = \{v_2, v_4, v_6, v_{10}, v_{12}, v_{14}, v_{16}, v_{21}, v_{22}\}$ for $n = 23, 24$;

respectively.

Hence $\gamma_t(G) = 7$ for $n = 19$ and $\gamma_t(G) = 8$ for $n = 20, 21, 22, 23, 24$.

Thus $\gamma_t(G) = 3$ for $n = 7$

$= 4$ for $n = 8, 9, 10, 11, 12$

$= 5$ for $n = 13$

$= 6$ for $n = 14, 15, 16, 17, 18$

$= 7$ for $n = 19$

$= 8$ for $n = 20, 21, 22, 23, 24$.

Hence we get that the general form of total dominating sets of $G$ are $TD = \{v_2, v_4, \ldots, v_{n-9}, v_{n-4}, v_{n-3}, v_{n-1}\}$ for $n = 7, 13, 19, \ldots$

$TD = \{v_2, v_4, \ldots, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}\}$ for $n = 8, 14, 20, \ldots$

$TD = \{v_2, v_4, \ldots, v_{n-6}, v_{n-5}, v_{n-3}, v_{n-2}\}$ for $n = 9, 15, 21, \ldots$

Thus $TD = \{v_2, v_4, \ldots, v_{n-7}, v_{n-6}, v_{n-3}, v_{n-1}\}$ for $n = 10, 16, 22, \ldots$

$TD = \{v_2, v_4, \ldots, v_{n-8}, v_{n-7}, v_{n-2}, v_{n-1}\}$ for $n = 11, 17, 23, \ldots$

$TD = \{v_2, v_4, \ldots, v_{n-9}, v_{n-8}, v_{n-3}, v_{n-2}\}$ for $n = 12, 18, 24, \ldots$

and so on.

Thus $\gamma_t(G) = 2k + 1$ for $n = 6k + 1$.

$= 2k + 2$ for $n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$

where $k = 1, 2, 3, \ldots$ respectively.

Theorem 4.2: Let $G$ be the interval graph with $n$ vertices and no isolated vertices, where $2 \leq n \leq 7$. Then $\gamma_t(G) = 2$.

Proof: Let $G$ be the interval graph with $n$ vertices and no isolated vertices, where $2 \leq n \leq 7$.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the vertices of $G$.

Then it is clear that $\{v_2, v_4\}$ is the total dominating set when $n = 3$ and $\{v_2, v_4\}$ is the total dominating set when $n = 4, 5, 6$.

That is $\gamma_t(G) = 2$.

Theorem 4.3: Let $G$ be the interval graph with $n$ vertices and no isolated vertices, where $n \geq 7$. Then the total Roman domination number of $G$ is $\gamma_{rR}(G) = 4k + 2$ for $n = 6k + 1$

$= 4k + 4$ for $n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$

where $k = 1, 2, 3, \ldots$ respectively.

Proof: Let $G$ be the interval graph with $n$ vertices and no isolated vertices, where $n \geq 7$.

Let the vertex set $G$ be $\{v_1, v_2, v_3, v_4, \ldots, v_n\}$.

Case 1: Suppose $n = 6k + 1$, where $k = 1, 2, 3, \ldots$

Let $f : V \to \{0, 1, 2\}$ and let $(V_0, V_1, V_2)$ be the ordered partition of $V$ induced by $f$ where $V_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2$. Then there exist a $1-1$ correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered pairs $(V_0, V_1, V_2)$ of $V$. Thus we write $f = (V_0, V_1, V_2)$.

Let $f_1 = \emptyset$;

$V_2 = \{v_3, v_4, \ldots, v_{n-9}, v_{n-4}, v_{n-3}, v_{n-1}\}$;

$V_{0} = V - V_2 = \{v_1, v_2, v_5, \ldots, v_{n-5}, v_{n-2}, v_n\}$.

By Theorem 4.1, we see that $V_2$ is a minimum total dominating set of $G$. Further the set $V_0$ dominates $V_2$. In addition the induced sub graph on $V_1 \cup V_2$ is a sub graph of $G$ with no isolated vertices.

Therefore $f = (V_0, V_1, V_2)$ is a total Roman dominating function of $G$.

Now $|V_2| = 2k + 1, |V_1| = 0, |V_0| = n - (2k + 1)$.

Therefore $\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v)$

$= 2(k + 1) = 4k + 2$.

Let $g = (V_{0}', V_1', V_2')$ be a total Roman dominating function of $G$, where $V_2'$ dominates $V_0$. Then
\[ g(V) = \sum_{v \in V} g(v) = \sum_{v \in V_0} g(v) + \sum_{v \in V_1} g(v) + \sum_{v \in V_2} g(v) = |V_1'| + 2|V_2'| \]

Since \( V_2 \) is a minimum dominating set of \( G \), we have \( |V_2| \leq |V_2'| \). Further \( |V_2'| > |V_2| \), since \( |V_2'| = 1 \). This implies that \( g(V) = |V_1'| + 2|V_2'| > |V_1| + 2|V_2| = f(V) \). Thus \( f(V) \) is the minimum weight of \( G \), where \( f(V_0, V_1, V_2) \) is a total Roman dominating function.

Therefore \( \gamma_{tr}(G) = 4k + 2 \).

**Case 2:** Suppose \( n = 6k + 2 \), where \( k = 1, 2, 3 \ldots \).

Now we proceed as in Case 1.

Let \( V_1 = \{ \emptyset \} \).

\[ V_2 = \{ v_3, v_4, \ldots \ldots \ldots , v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1} \} \].

\[ V_0 = V - V_2 = \{ v_1, v_2, v_5, \ldots \ldots \ldots , v_{n-6}, v_{n-3}, v_n \} \].

Clearly \( V_2 \) is a minimum total dominating set of \( G \). Here we observe that the set \( V_2 \) dominates \( V_0 \).

Therefore \( f = (V_0, V_1, V_2) \) becomes a total Roman dominating function of \( G \).

Now \( |V_2| = 2k + 2, |V_0| = 0, |V_1| = n - (2k + 2) \).

Therefore \[ \sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v) = 2(2k + 2) = 4k + 4 \].

Let \( g = (V_0, V_1, V_2) \) be a total Roman dominating function of \( G \). Then we can show as in Case 1, that \( f(V) \) is a minimum weight of \( G \) for the total Roman dominating function \( f(V_0, V_1, V_2) \).

Thus \( \gamma_{tr}(G) = 4k + 4 \).

**Case 3:** Suppose \( n = 6k + 3 \), where \( k = 1, 2, 3 \ldots \).

Now we proceed as in Case 1.

Let \( V_1 = \{ \emptyset \} \).

\[ V_2 = \{ v_3, v_4, \ldots \ldots \ldots , v_{n-6}, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1} \} \].

\[ V_0 = V - V_2 = \{ v_1, v_2, v_5, \ldots \ldots \ldots , v_{n-4}, v_{n-3}, v_n \} \].

Here \( V_2 \) is a minimum total dominating set of \( G \). Here we observe that the set \( V_2 \) dominates \( V_0 \).

Therefore \( f = (V_0, V_1, V_2) \) becomes a total Roman dominating function of \( G \).

Now \( |V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2) \).

Therefore \[ \sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v) = 2(2k + 2) = 4k + 4 \].

Let \( g = (V_0, V_1, V_2) \) be a total Roman dominating function of \( G \). Then we can show as in Case 1, that \( f(V) \) is a minimum weight of \( G \) for the total Roman dominating function \( f(V_0, V_1, V_2) \).

Thus \( \gamma_{tr}(G) = 4k + 4 \).

**Case 4:** Suppose \( n = 6k + 4 \), where \( k = 1, 2, 3 \ldots \).

Now we proceed as in Case 1.
Let \( g = (V'_0, V'_1, V'_2) \) be a total Roman dominating function of \( G \). Then we can show as in Case 1, that \( f(V) \) is a minimum weight of \( G \) for the total Roman dominating function \( f(V_0, V_1, V_2) \).

Thus \( \gamma_{TR}(G) = 4k + 4 \).

**Theorem 4.4:** Let \( G \) be the interval graph of with \( n \) vertices and no isolated vertices, where \( 2 < n < 7 \). Then the total Roman domination number is

\[
\gamma_{TR}(G) = 3 \text{ for } n = 3, 4
\]

\[
= 4 \text{ for } n = 5, 6.
\]

**Proof:** Let \( G \) be the interval graph of with \( n \) vertices and no isolated vertices, where \( 2 < n < 7 \). Let \( \{v_1, v_2, v_3, v_4, v_5, v_6\} \) be the vertices of \( G \).

**Case 1:** Suppose \( n = 3 \). Let \( v_1, v_2, v_3 \) be the vertices of \( G \).

Let \( V_1 = \{v_3\}; \quad V_2 = \{v_2\}; \quad V_0 = V - (V_1 \cup V_2) = \{v_1\} \).

We observe that \( V_1 \cup V_2 \) is a minimum total dominating set of \( G \) and the set \( V_2 \) dominates \( V_0 \). In addition the induced sub graph on \( V_1 \cup V_2 \) is a sub graph of \( G \) with no isolated vertices.

Therefore \( f = (V_0, V_1, V_2) \) is a total Roman dominating function of \( G \).

Then

\[
\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v).
\]

\[= 0 + 1 + 2 = 3 \]

Thus \( \gamma_{TR}(G) = 3 \).

**Case 2:** Suppose \( n = 4 \). Let \( v_1, v_2, v_3, v_4 \) be the vertices of \( G \).

Let \( V_1 = \{v_4\}; \quad V_2 = \{v_3\}; \quad V_0 = V - (V_1 \cup V_2) = \{v_1, v_2\} \).

Clearly \( V_1 \cup V_2 \) is a minimum total dominating set of \( G \) and the set \( V_2 \) dominates \( V_0 \). Now we proceed as in Case 1, and hence we have \( \gamma_{TR}(G) = 3 \).

**Case 3:** Suppose \( n = 5 \). Let \( v_1, v_2, v_3, v_4, v_5 \) be the vertices of \( G \).

Let \( V_1 = \{\emptyset\}; \quad V_2 = \{v_3, v_4\}; \quad V_0 = V - (V_1 \cup V_2) = \{v_1, v_2, v_5\} \).

We observe that \( V_2 \) is a minimum total dominating set of \( G \) and the set \( V_2 \) dominates \( V_0 \).

Therefore \( f = (V_0, V_1, V_2) \) is a total Roman dominating function of \( G \).

Then

\[
\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v).
\]

\[= 0 + 2 + 2 = 4 \]

Thus \( \gamma_{TR}(G) = 4 \).

**Case 4:** Suppose \( n = 6 \). Let \( v_1, v_2, v_3, v_4, v_5, v_6 \) be the vertices of \( G \).

Let \( V_1 = \{\emptyset\}; \quad V_2 = \{v_3, v_4\}; \quad V_0 = V - (V_2) = \{v_1, v_2, v_5, v_6\} \).

Clearly \( V_2 \) is a minimum total dominating set of \( G \) and the set \( V_2 \) dominates \( V_0 \). Now we proceed as in Case 4, and hence we have \( \gamma_{TR}(G) = 4 \).

**Theorem 4.5:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( 2 < n < 5 \). Then \( \gamma_{TR}(G) = \gamma_1(G) + 1 \).

**Proof:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( 2 < n < 5 \).

Then it is clear that when \( n = 3, 4 \), from Theorem 4.4 and Theorem 4.2 we get

\[
\gamma_{TR}(G) = 3 = 2 + 1 = \gamma_1(G) + 1.
\]

Thus \( \gamma_{TR}(G) = \gamma_1(G) + 1 \).

**Theorem 4.6:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices. If \( \gamma(G) = \gamma_1(G) \), then \( G \) is a total Roman graph.

**Proof:** Let \( G \) be the interval graph of with \( n \) vertices and no isolated vertices.

Suppose \( n = 6 \).

Then we have \( \gamma(G) = 2 \) and \( \gamma_1(G) = 2 \).

Thus \( \gamma(G) = \gamma_1(G) \).

Therefore \( G \) is a total Roman graph.

**Theorem 4.7:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( n \geq 7 \). Then \( G \) is a total Roman graph.

**Proof:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( n \geq 7 \).

Then by Theorem 4.3, we have the total Roman domination number is

\[
\gamma_{TR}(G) = 4k + 2 \text{ for } n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6,
\]

where \( k = 1, 2, 3, \ldots \) respectively.

**Case 1:** Suppose \( n = 6k + 1 \), and \( k = 1, 2, 3, \ldots \) respectively.

Then by Theorem 4.3, we have the total Roman domination number is

\[
\gamma_{TR}(G) = 4k + 4 \text{ for } n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6,
\]

where \( k = 1, 2, 3, \ldots \) respectively.

**Theorem 4.8:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( n \geq 7 \). Then \( G \) is a total Roman graph if and only if there exist a \( \gamma_{TR} \)-function \( f = (V_0, V_1, V_2) \) with \( |V_2| = 0 \).
**Proof:** Let \( G \) be the interval graph with \( n \) vertices and no isolated vertices, where \( n \geq 7 \). Suppose \( G \) is a total Roman graph. Let \( f = (V_0, V_1, V_2) \) be a \( \gamma_{\text{TR}} \)-function of \( G \). Then we know that \( V_2 \) dominates \( V_0 \) and \( V_1 \cup V_2 \) dominates \( V \). In addition the induced sub graph \( V_1 \cup V_2 \) is a sub graph of \( G \) with no isolated vertices. Hence \( \gamma_r(G) \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{\text{TR}}(G) \). But \( G \) is a total Roman graph. So \( \gamma_{\text{TR}}(G) = 2 \gamma_r(G) \). Then it follows that \( |V_1| = 0 \), which establishes Theorem 4.3.

Conversely, suppose there is a \( \gamma_{\text{TR}} \)-function \( f = (V_0, V_1, V_2) \) of \( G \) such that \( |V_0| = 0 \). By the definition of \( \gamma_{\text{TR}} \)-function, we have \( V_1 \cup V_2 \) dominates \( V \) and since \( |V_1| = 0 \), it follows that \( V_2 \) dominates \( V \). In addition the induced sub graph \( V_1 \cup V_2 \) is a sub graph of \( G \) with no isolated vertices. As \( V_2 \) is a minimum total dominating set, we have \( \gamma_r(G) = |V_2| \). By the definition of \( \gamma_{\text{TR}} \)-function we have \( \gamma_{\text{TR}}(G) = |V_1| + 2|V_2| = 0 + 2|V_2| = 2 \gamma_r(G) \).

Hence \( G \) is a total Roman graph, which also establishes Theorem 4.3.

5. **ILLUSTRATIONS**

**Illustration 1:** \( n = 7 \)

**Illustration 2:** \( n = 9 \)

\[ TD = \{v_3, v_4, v_6\} \text{ and } \gamma_r(G) = 3. \]

\[ V_1 = \emptyset; \quad V_2 = \{v_3, v_4, v_6\}; \quad V_0 = V - \{V_2\} = \{v_1, v_2, v_5, v_7\} \]

\[ \sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2 \times 3 = 6 = f(V) \]

Therefore \( \gamma_{\text{TR}}(G) = 6. \)

**Illustration 2:**

\[ TD = \{v_3, v_4, v_6, v_8\} \text{ and } \gamma_r(G) = 4. \]

\[ V_1 = \emptyset; \quad V_2 = \{v_3, v_4, v_6, v_8\}; \quad V_0 = V - \{V_2\} = \{v_1, v_2, v_5, v_7, v_9\} \]

\[ \sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2 \times 4 = 8 = f(V) \]

Therefore \( \gamma_{\text{TR}}(G) = 8. \)
5. REFERENCES


