International Journal of Applied Information Systems (IJAIS) – ISSN : 2249-0868
Foundation of Computer Science FCS, New York, USA
Volume 12– No.27, February 2020 – www.ijais.org

Weak Domination in Block Graphs

M. H. Muddebihal
Professor
Department of Mathematics
Gulbarga University,
Kalaburagi-585106,
Karnataka, India

Geetadevi Baburao
Research Scholar
Department of Mathematics
Gulbarga University,
Kalaburagi-585106,
Karnataka, India

ABSTRACT
For any graph $G=(V,\ E)$, the block graph $B(G)$ is a graph whose set of vertices is the union of set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. For any two adjacent vertices u and v we say that v weakly dominates u if $\deg(v) \leq \deg(u)$. A dominating set D of a graph $B(G)$ is a weak block dominating set of $B(G)$, if every vertex in $V[B(G)]-D$ is weakly dominated by at least one vertex in D. A weak domination number of a block graph $B(G)$ is the minimum cardinality of a weak dominating set of $B(G)$. In this paper, we study a graph theoretic properties of $\gamma_{WB}(G)$ and many bounds were obtained in terms of elements of G and the relationship with other domination parameters were found.

Keywords
Dominating set; Strong split domination; Weak domination; Perfect domination; Weak block domination.

Subject classified number: AMS-05C69, 05C70.

1. INTRODUCTION
We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The open neighborhood of a vertex

$v \in V$ *is* $N(v)=\{u \in V | uv \in E\}$ and the closed neighborhood is $N[V]=N(V) \cup \{v\}$. For a subset $S \subseteq V$, the open neighborhood is $N(S)=\cup_{v \in S} N(v)$ and the closed neighborhood is $N[S]=N(S) \cup S$. If v is vertex of V, then the degree of v denoted by $\deg(v)$, is the cardinality of its open neighborhood. By $\Delta(G)$=Δ we denote the maximum degree of a graph G. The minimum distance between any two furthest vertices of a connected graph G is called the diameter of G and is denoted by diam(G). In literature, the concept of graph theory terminology not presented here can be found in Harary [6].

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V-S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$.

Further, A set F of edges is an edge dominating set, if for every edge $e \in E-F$ there exist an edge $f \in F$ such that e and f have a vertex in common. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G see [15].

A dominating set $S \subseteq V(G)$ is called the total dominating set, if for every vertex $v \epsilon V$, there exist a vertex $u \in S$, $u \neq v$ such that u is adjacent to v. The total dominationnumber of G is denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. This was introduced by Cockayne [2].

In [12] Hadetniemi and Laskar defined a connected dominating set. A dominating set $S \subseteq V(G)$ is connected dominating set, if the induced subgraph $< S >$ is connected. The connected domination number $\gamma_c(G)$ of a graph G is the minimum cardinality of connected dominating set of G.

An independent domination of a graph G was studied by Allan [1]. A dominating set D of a graph $G=(V,E)$ is an independent dominating set, if the induced subgrapg $< D >$ has no edges. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

A dominating set $S \subseteq V(G)$ is called the double dominating set for G, if each vertex in V is dominated by at least two vertices in S. The double domination number $\gamma_{dd}(G)$ of G is the minimum cardinality of a double dominating set of G see[7].

Analogously, a set $S \subseteq V(G)$ is a Restrained dominating set of G, if every vertex in $V-S$ is adjacent to a vertex in S and another vertex in $V-S$. The Restrained domination number of a graph G is denoted by $\gamma_{Res}(G)$ is the minimum cardinality of a Restrained dominating set in G see in [5].

A dominating set $S \subseteq V(G)$ is called the Perfect dominating set of G, if each $u \epsilon V(G)-S$ is dominated by exactly one element of S. The Perfect domination number of G, denoted by $\gamma_p(G)$ is the minimum cardinality of a Perfect dominating set of G. This was introduced by Cockayne [4].

The lict graph $n(G)$ of a graph G is the graph whose set of vertices is the union of set of edges and the set of cutvertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident formed in [14].

A set $S \subseteq V(G)$ is a cototal dominating set, if the induced subgraph $< V-S >$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ is the minimum cardinality of a cototal dominating set of G defined in [13].
A dominating set $S \subseteq V(G)$ is a split dominating set, if the induced subgraph $< V-D >$ is disconnected. The split domination number $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set in [13].
A dominating set $D \subseteq V(G)$ is the strong split dominating set, if the induced subgraph $< V-D >$ is totally

disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ of a graph G is the minimum cardinality of a strong split dominating set of G see [13].

In [13] a dominating set $D \subseteq V(G)$ is a nonsplit dominating set, if the induced subgraph $< V - D >$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of a graph G is the minimum cardinality of a nonsplit dominating set.

A dominating set $D \subseteq V(G)$ is a strong nonsplit dominating set, if the induced subgraph $< V - D >$ is complete. The strong nonsplit domination number $\gamma_{sns}(G)$ of G is the minimum cardinality of a strong nonsplit dominating set formed in [13].

In [26], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a weak dominating set (WDS) if every vertex $u \epsilon V - S$ is adjacent to a vertex $v \epsilon D$, where $\deg(u) \geq \deg(v)$. The subset D is a strong dominating set (SDS) if every vertex $v \epsilon V - S$ is adjacent to a vertex $u \epsilon D$, where $deg(u) \geq deg(v)$. The weak (strong, respectively) domination number $\gamma_w(T)(\gamma_s(T)$, respectively) is the minimum cardinality of a WDS(a SDS, respectively) of G. Strong and weak domination have been studied for example in $[8, 9, 16, 23, 24, 25]$. For more details on domination in graphs and its variation see the two books $[10, 11]$. Farther domination related graph valued functions has been studied in $[17, 18, 19, 20, 21, 22]$.

In this paper we initiate the study of weak block domination in graphs.

2. RESULTS

We begin by the following straight forward observation.

Observation 1: Every weak block dominating set of a graph G contains all the end vertices of G.

Next result is a lower bound on the weak block domination number for trees.

Theorem 2.1: For any nontrivial tree T, $\gamma_{wb}(T) \geq \gamma(T)$.

Proof: Let $D = \{v_1, v_2, v_3, \dots, v_m\} \subseteq V(T)$ such that $N[D] = V(T)$. Then D itself is a dominating set of T. Let $A = \{e_1, e_2, e_3, \dots, e_m\}$ be the edge set of T and $B = \{v_1, v_2, v_3, \dots, v_m\} = V[B(T)]$ be the set of vertices corresponding to the edges of A and has no end vertices. Now we consider a set $B_1 \subseteq B$ be the set of minimum degree vertices which are nonend vertices in $B(T)$. Suppose $B_2 \subseteq B_1$ such that $N[B_2] = V[B(T)]$. Then B_2 is dominating set of $B(T)$. Which is also a $\gamma_{WB} - set$. Hence $|B_2| \geq |D|$ gives required result.

Further, if $B(T)$ has end vertices then, $C = \{v_1, v_2, v_3, \dots, v_m\}$ be the set of end vertices in $B(T)$. Since B_2 is $\gamma_{WB} - set$, by the definition it is also true that $\{B_2 \cup C\}$ forms a $\gamma_{WB} - set$. Hence, again $|\{B_2 \cup C\}| \geq |D|$ and gives $\gamma_{WB}(T) \geq \gamma(T)$.

Now we establish the relationship between domination number, strong split domination with weak block domination number.

Theorem 2.2: For any tree T, $\gamma_{WB}(T) \leq \gamma_{ss}(T) + \gamma(T) + 1$.

Proof: Let $D = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the set of nonend vertices such that $N[D] = V[T]$. Then D is a minimal dominating set of T.

If for every $v_i \epsilon V - D$, with $\deg(v_i) = 0$ and $< V - D >$ has at least two vertices, then D is a $\gamma_{ss} - set\ of\ T$. Otherwise if there exists a vertex set $H = \{v_1, v_2, v_3, \dots, v_k\}$ and every vertex of H is incident to at least one edge, where $H \in V - D$. Now consider $H_1 \subseteq H\ \forall\ v_i \in < H - H_1 >$, $\deg(v_i) = 0\ and < V - \{D \cup H_1\} >$ has at least two isolated vertices. Clearly $\{D \cup H_1\}$ is a $\gamma_{ss} - set$ of T.

Let $A = \{b_1, b_2, b_3, \dots, b_n\}$ be the set of blocks in T. Then $A_1 = \{v_1, v_2, v_3, \dots, v_n\} = V[B(T)]$ corresponding to the blocks of A. Consider J as a dominating set of $B(T)$. Suppose $\forall\ v_i \in J$, $\deg(v_i) \leq \deg(v_j)$, $\forall\ v_j \in V[B(T)] - J$. Then J itself is a weak dominating set of $B(T)$. If not, then there exists a set $S \subseteq V[B(T)] - J$ such that $\deg(v_k) < \deg(v_i)$, $\forall\ v_k \in S$, hence the set $J \cup \{S\}$ gives a weak dominating set of $B(T)$. So that $|J \cup \{S\}| \leq |D \cup H_1| + |D| + 1$, gives $\gamma_{WB}(T) \leq \gamma_{SS}(T) + \gamma(T) + 1$.

The following result gives an upper bound on $\gamma_{WB}(T)$ in terms of vertices and maximum degree of G.

Theorem 2.3: for any nontrivial (p, q) tree T, $\gamma_{WB}(T) \leq p - \Delta(T)$.

Proof: Let $E = \{v_1, v_2, v_3, \dots, v_i\}$ be the edge set of T. Then $D = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vertices in $B(T)$ corresponding to the edges of E. Let $D_1 \subseteq D$ be the set of all end vertices. Suppose $D_2 \subseteq D$ be the set of vertices with minimum degree which are adjacent to the cut vertices of $B(T)$ and covers all the vertices of $B(T)$. Then D_2 is minimal dominating set of $B(T)$.

If $D_1 \neq \emptyset$, then $D_2 \cup D_1$ forms a $\gamma_{WB} - set$. Otherwise D_2 itself is a $\gamma_{WB} - set$. Since for any tree T, there exist at least one vertex v, $\deg(v) = \Delta(T)$ and $p = V(T)$. It follows that $|D_2 \cup D_1|$ or $|D_2| \leq |V(T)| - \Delta(T)$. Hence $\gamma_{WB}(T) \leq p - \Delta(T)$.

In the following theorem we establish the relation between $\gamma_{WB}(T)$, $\gamma_{cot}(T)$ and $diam(T)$.

Theorem 2.4: For any non trivial tree T with $n \geq 2$ blocks, $\gamma_{WB}(T) \leq \gamma_{cot}(T) + diam(T) + 1$.

Proof: Let $J = \{e_1, e_2, e_3, \dots, e_n\} \subseteq E(T)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u, v \epsilon V(T)$ such that $dist(u, v) = diam(T)$. Let $D = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the minimum set of vertices which covers all the vertices in T. Suppose the subgraph $< V(T) - D >$ has no isolated vertex then D itself is a $\gamma_{cot} - set\ of\ T$. Otherwise if there exist a set $H = \{v_1, v_2, v_3, \dots, v_j\} \subseteq V(T) - D$ with $\deg(v_i) = 0$, $1 \leq i \leq j$. Now we make $\deg(v_i) = 1$ by joining vertices $\{v_k\} \subseteq V(T) - D$ and $N(v_i) \epsilon \{v_k\}$. Clearly $D_1 = D \cup H - \{v_k\}$ forms a minimal cototal dominating set of T.

Suppose $B = \{b_1, b_2, b, \dots, b_m\}$ be the set of vertices of block graph $B(T)$. Suppose $B_1 \subseteq B\ \forall\ v_i \in B_1$ has $\deg\ (v_i) < \Delta[B(T)]$ and $N[B_1] = V[B(T)]$ and $\deg(v_i) \leq \deg(v_j)$, $\forall\ v_j \in V[B(T)] - B_1$. Then B_1 is a $\gamma_{WB} - set$. It follows that $|B_1| \leq |D_1| + diam(T) + 1$ which gives $\gamma_{WB}(T) \leq \gamma_{\cot}(T) + diam(T) + 1$.

In the following theorem we develop the relation between $\gamma_{WB}(T)$, $\gamma_{Res}(T)$ and $diam(T)$.

Theorem 2.5: For any non trivial tree T with $n \geq 2$ $blocks$, $\gamma_{WB}(T) \leq \gamma_{Res}(T) + diam(T) + 1$.

Proof: Let $F = \{e_1, e_2,, e_3,, \dots, e_n\} \subseteq E(T)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u,\ v \in V(T)$ such that $dist(u, v) = diam(T)$.

Suppose $B = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V(T)$ be the set of all end vertices. Suppose $S = B \cup B'$, where $B' \subseteq V(T) - B$ be the set of vertices covering all the vertices with $diam(u, v) \geq 3$, $\forall\, u \in B$, $\forall\, v \in B'$ or for every vertex $w \in V(T) - S$, there exists at least one vertex $z \in V(T) - S$ such that wz is an edge in $V(T) - S$. Clearly S forms a minimal $\gamma_{Res} - set$ of T.

Let $E = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of edges in T. Then $A = \{v_1, v_2, v_3, \dots, v_m\} = V[B(T)]$ corresponding to the edges of E. Suppose $A_1 \subseteq A$, $\forall\ v_j \in A_1, \deg(v_j) = 1$ and $A_2 \subseteq A$ be the set of minimum degree vertices which are adjacent to a cut vertex of $B(T)$, since each block of $B(T)$ is complete and covers all the vertices of $B(T)$. Then $\{A_1 \cup A_2\}$ is a minimal weak dominating set of $B(T)$. Clearly $|A_1 \cup A_2| \leq |S| + |F| + 1$. Hence $\gamma_{WB}(T) \leq \gamma_{Res}(T) + diam(T) + 1$.

Roman domination: The concept of Roman domination function (RDF) was introduced by Cockayne [3]. A Roman domination function of a graph $G = (V, E)$ is a function $f: V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function in G is the value of $f(v) = \sum_{u \in v} f(u)$. The Roman domination number of a graph G is denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G.

Further, we relates $\gamma_{WB}(T)$ with Roman domination number and domination number.

Theorem 2.6: For any $(p,\ q)$ tree T, $\gamma_{WB}(T) \leq \gamma_R(T) + \gamma(T) + 1$.

Proof: Let $S = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the set of vertices with $\deg(v_i) \geq 2$, $\forall\, v_i \in S$, $1 \leq i \leq n$. Further, let there exist a set $S_1 \subseteq S$ of vertices with diam $(u, v) \geq 3$, $\forall\, u, v \in S_1$ which covers all the vertices in T. Clearly, S_1 forms a dominating set of T. Otherwise, if $diam(u, v) < 3$, then there exists at least one vertex $x \notin S_1$ such that $S' = S_1 \cup \{x\}$ forms a minimal $\gamma - set$ set of T. Suppose $f: V(T) \to \{0, 1, 2\}$ and partition the vertex set $V(T)$ into (V_0, V_1, V_2) induced by f with $|V_i| = n_i$ for $i = 0, 1, 2$. Suppose the set V_2 dominates V_o, then $H = V_1 \cup V_2$ forms a minimal Roman dominating set of T. Suppose D be a $\gamma_{WB} - set$ of tree T and assume $E = \{e_1, e_2, e_3, \dots, e_n\} = E(T)$. Let $E_1 \subseteq E$ be the minimum degree edges in T and $E_2 \subseteq E$ be the maximum degree edges in T. If $E_2' \subseteq E_2$ and since $\{E\} = V[B(T)]$, then $\{E_1 \cup E_2'\} \in V[B(T)]$. So that $\forall\, v_i \in V[B(T)] - \{E_1 \cup E_2'\}$ is adjacent to at least one vertex of $\{E_1 \cup E_2'\}$. Further if deg $(v_i) \in V[B(T)] - \{E_1 \cup E_2'\}$ is greater than or equal to $\deg(v_j) \in \{E_1 \cup E_2'\}$. Clearly $\{E_1 \cup E_2'\} = D$. Hence $|\{E_1 \cup E_2'\}| \leq |H| + |S'| + 1$ and implies $\gamma_{WB}(T) \leq \gamma_R(T) + \gamma(T) + 1$.

The following theorem gives upper bound for edges of tree in terms of $\gamma_p(T)$ and $\gamma_{WB}(T)$.

Theorem 2.7: For any non trivial tree T, then $\gamma_p(T) + \gamma_{WB}(T) \leq q$, $T \neq p_4$.

Proof: Let $D = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ such that $N(v_i) \cap N(v_j) = \emptyset$, $\forall\, v_i, v_j \in D$. Let $S = \{v_1, v_2, v_3, \dots, v_m\} \subseteq V(T) - D$ be the minimal set of vertices which covers all the vertices in T. Suppose every vertex $v_k \in V(T) - S$ is adjacent to exactly one vertex of S. Then S is a $\gamma_p - set$ of T.

Let $H = \{e_1, e_2, e_3, \dots, e_m\}$ be the edge set of T. In $B(T)$, $M = \{v_1, v_2, v_3, \dots, v_m\} = V[B(T)]$ corresponding to the edges of H. Now we consider a set $M_1 \subseteq M$ be the set of end vertices in $B(T)$. Let $M_2 \subseteq M$ be the set of minimum degree vertices which are nonend vertices in $B(T)$. Suppose $M_3 \subseteq M_2$ and $N[M_1 \cup M_3] = V[B(T)]$. Then $S = \{M_1 \cup M_3\}$ is a $\gamma_{WB} - set$ of $B(T)$. Thus $|S| + |M_1 \cup M_3| \leq |E|$ which gives $\gamma_p(T) + \gamma_{WB}(T) \leq q$.

The following theorem gives an upper bound for $\gamma_{WB}(T)$.

Theorem 2.8: For any non trivial tree T, $\gamma_{WB}(T) \leq \gamma_n(T) + \gamma_c(T) - 1$.

Proof: Let $V_l = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the set of all nonend vertices in T. Suppose there exists a minimal set of vertices $S = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V_l$. Such that $N[v_i] = V(T)$, $\forall\, v_i \in S$, $1 \leq i \leq k$. Then S forms a minimal dominating set of T.

Further, if the subgraph $< S >$ has exactly one component, then S itself is a connected dominating set of T. Suppose S has more than one component, then attach the minimum set of vertices S' of $V_l - S$, which are in every $u - w$ path, $\forall\, w \in S$, $\forall\, u \in V_l - S$ gives a single component. $S_1 = S \cup S'$. Clearly S_1 forms a minimal $\gamma_c - set$ of T.

Let $E_1 = \{e_1, e_2, e_3, \dots, e_n\} \subseteq E(T)$, deg $(e_i) \geq 3$, $1 \leq i \leq n$ and $E_2 = E(T) - E_1$. Since $V[n(T)] = E_1 \cup E_2 \cup C$, $\forall\, v_i \in C$ is a cutvertices of T. Then there exists a minimal set $E_1' \subseteq E_1$ which covers all the vertices of $n(T)$. Clearly E_1' forms a minimal $\gamma_n - set$ of T.

Now we consider the tree T such that each block of T is an edge. Let $B = \{B_1, B_2, B_3, \dots, B_k\}$ be the set of blocks in T. Suppose $F = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V[B(T)]$ be the set of vertices with $\deg(v_j) \geq 1$. Suppose there exists a vertex set $D \subseteq F$ with $N[D] = V[B(T)]$ and if $|\deg(x) - \deg(y)| \leq 1$. $\forall\, x \in V[B(T)] - D$, $\forall\, y \in D$. Then D forms a weak block dominating set of T. Otherwise there exists at least one vertex $\{w\} \subseteq F$ where $\{w\} \notin D$ such that $D \cup \{w\}$ forms a minimal $\gamma_{WB} - set$. It follows that $|D| \leq |E_1'| + |S_1| - 1$. Clearly $\gamma_{WB}(T) \leq \gamma_n(T) + \gamma_c(T) - 1$.

Theorem 2.9: For any tree T, $\gamma_{WB}(T) \leq \gamma'(T) + \gamma_t(T) + \Delta(T) + 1$.

Proof: Let $F^{'} = \{e_1, e_2, e_3, \dots, e_m\}$ be the set of all end edges in T. Suppose $E - F^{'} = I$, then $S \subseteq I$ forms an $\gamma^{'} - set$ of T. Further, if $E - F^{'} = \emptyset$, then there exists at least one edge $\{e\} \in F^{'}$ such that $S = \{e\}$ forms a minimal edge dominating set of T. If $A = \{v_1, v_2, v_3, \dots, v_n\}$ be the minimal set of vertices with $\deg(v_i) \geq 2$, $1 \leq i \leq n$ and $N[v_i] = V[T]$. And if the subgraph $< A >$ has no isolated vertex, then A itself is a minimal total dominating set of T. Otherwise, if there exists a vertex $x \in A$, $\deg(x) = 0$, then attach a vertex which is $N(x)$ and $< A \cup \{x\} >$ has no isolates. Then $A \cup \{x\}$ is a minimal total dominating set of T.

Let $J = \{e_1, e_2, e_3, \dots, e_m\}$ be the edge set of T. In $B(T)$, $D = \{v_1, v_2, v_3, \dots, v_m\} = V[B(T)]$ corresponding to the edges of J. Suppose $D_1 \subseteq D$ be the end vertices of $B(T)$. Let $D_2 \subseteq D$ be the nonend vertices with minimum degree and $N[\{D_1 \cup D_2\}] = V[B(T)]$ with the property that $\deg(u) \geq \deg(v)$, $\forall\, u \in V[B(T)] - \{D_1 \cup D_2\}$ and $v \in \{D_1 \cup D_2\}$. Hence $\{D_1 \cup D_2\}$ forms a minimal weak block dominating set of T. Since for any tree T, there exists at least one vertex $v \in V(T)$ with $\deg(v) = \Delta(T)$. Clearly $|D_1 \cup D_2| \leq |S| + |A + \{x\}| + \Delta(T) + 1$ which gives $\gamma_{WB}(T) \leq \gamma^{'}(T) + \gamma_t(T) + \Delta(T) + 1$.

Theorem2.10: For any non trivial T, $\gamma_{WB}(T) \leq \gamma_s(T) + \alpha_o(T) + \left\lceil \frac{C}{2} \right\rceil + 2$. where C is the number of cut vertices in T.

Proof: Let $B = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the minimal set of vertices with $dist(u, v) \geq 2$ for all $u, v \in B$ covers all the edges in T. Clearly, $|B| = \alpha_0(T)$. Let $D \subseteq V(T)$ be the set of vertices such that $N[D] = V(T)$ and if the subgraph $< V(T) - D >$ contains more than one component, then D forms a split dominating set of T. Otherwise there exists at least one vertex $\{u\} \in V(T) - D$ such that $< V(T) - D - \{u\} >$ yields more than one component. Clearly, $D \cup \{u\}$ forms a minimal $\gamma_s - set$ of T.

Let $A = \{e_1, e_2, e_3, \dots, e_n\}$ be the edge set of T. Let $H = \{u_1, u_2, u_3, \dots, u_n\} = V[B(T)]$ be the set of vertices corresponding to the edges of A. Let $J \subseteq H$ be the set of vertices with $\deg(w) \geq 1$ for every $w \in J$ such that $N[J] = V[B(T)]$ and if $\forall\, v_i \in J$ has degree at least 2 and $v_j \in V[B(T)] - J$ and $\deg(v_j) \geq \deg(v_i)$. Then J forms $\gamma_{WB} - set$. Let C be set of cut vertices which are nonend vertices in T which gives $|J| \leq |D \cup \{u\}| + |B| + \left\lceil \frac{C}{2} \right\rceil + 2$. Clearly $\gamma_{WB}(T) \leq \gamma_s(T) + \alpha_0(T) + \left\lceil \frac{C}{2} \right\rceil + 2$.

Theorem2.11: For any tree T, $\gamma_{WB}(T) \leq i(T) + \alpha_0(T) + \Delta(T) + m - 1$ where m is the number of support vertex in T.

Proof: Suppose $A = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V(T)$ be the set of vertices which covers all the vertices in T. Further, if the $< A >$ $\forall\, v_i \in A$, $\deg(v_i) = 0$, $1 \leq i \leq n$, then A itself is an independent dominating set of T. Otherwise $S = A^{'} \cup I$, where $A^{'} \subseteq A$ and $I \subseteq V(T) - A$ forms a minimal independent dominating set of T. Let $C = \{u_1, u_2, u_3, \dots, u_n\} \subseteq V(T)$ be the minimal set of vertices which covers all the edges in T then $|C| = \alpha_0(T)$.

Suppose $V = \{v_1, v_2, v_3, \dots, v_p\}$ be the set of vertices in T then there exists at least one vertex $v \in V$ such that $\deg(v) = \Delta(T)$.

Let $M = \{v_1, v_2, v_3, \dots, v_m\}$ be the set of all support vertices in T with $|M| = m$.

Suppose $S = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of edges in T. Then $H = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vertices in $B(T)$ corresponding to the edges of S. Let $H_1 = \{v_1, v_2, v_3, \dots, v_i\} \subseteq H$ be the set of cutvertices in $B(T)$, since $\deg(v_i) \geq \deg(v_j)$, $\forall\, v_i \in H_1$ and $\forall\, v_j \in H - H_1$. Let D be the weak dominating set of $B(T)$ such that $D \subseteq \{H - H_1\}$ and hence $|D| \leq |S| + |C| + |\deg(v)| + |M| - 1$ which gives $\gamma_{WB}(T) \leq i(T) + \alpha_0(T) + \Delta(T) + m - 1$.

We establish the following upper bound for $\gamma_{WB}(T)$.

Theorem 2.12: For any tree T, $\gamma_{WB}(T) \leq \gamma_W(T) + \Delta^{'}(T) + 2$.

Proof: Let $F = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V(T)$ be the set of vertices with $\deg(v_j) \geq 1$, $1 \leq j \leq k$. Suppose there exists a vertex set $D \subseteq F$ with $N[D] = V(T)$ and if $|\deg(x) - \deg(y)| \leq 1$, $\forall\, x \in [V(T)] - D$, $\forall\, y \in D$. Then D forms a weak dominating set in T. Otherwise there exists at least one vertex $\{w\} \in F$ with $\{w\} \notin D$ such that $D \cup \{w\}$ forms a minimal $\gamma_w - set$ in T.

Let $H = \{u_1, u_2, u_3, \dots, u_n\} = V[B(T)]$, suppose $S \subseteq H$ be the set of vertices with $\deg(w) \geq 1$ for every $w \in S$ such that $N[S] = V[B(T)]$ and if $\forall\, v_i \in S$ has degree at least 2 and $v_j \in V[B(T)] - S$ and $\deg(v_j) \geq \deg(v_i)$. Then S forms $\gamma_{WB} - set$. For any graph T, there exists at least one edge $e \in E(T)$ with $\deg(e) = \Delta^{'}(T)$. Clearly it follows that $|S| \leq |D \cup \{w\}| + |\deg(e)| + 2$ gives $\gamma_{WB}(T) \leq \gamma_W(T) + \Delta^{'}(T) + 2$.

Theorem 2.13: For any tree T, with $n \geq 2$ blocks then $\gamma_{WB}(T) \geq \left\lceil \frac{diam(T)+1}{3} \right\rceil$.

Proof: Let $V = \{v_1, v_2, v_3, \dots, v_j\}$ be the set of all vertices in T such that there exists 2 vertices $u, v \in V(T)$ and $dist(u, v)$ forms a diametral path in T. Clearly, $dist(u, v) = diam(T)$. Let $F = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of edges in T and $F \subseteq E(T)$. Then in $B(T)$, $D = \{v_1, v_2, v_3, \dots, v_n\}$ which corresponds to $\forall\, e_i \in F$. Let $\deg(e_i)$, $\forall\, e_i \in F$ and $\deg(e_j)$ $\forall\, e_j \in E(T) - F$ such that $\deg(e_i) \leq \deg(e_j)$. Suppose $D_1 = \{v_1, v_2, v_3, \dots, v_i\} \subseteq D$ and $N[D_1] = V[B(T)]$. Then D_1 forms a $\gamma_{WB} - set$. It follow that $|D_1| \geq \left\lceil \frac{diam(T)+1}{3} \right\rceil$, gives $\gamma_{WB}(T) \geq \left\lceil \frac{diam(T)+1}{3} \right\rceil$.

A relationship between weak block domination number and the edge covering number of T is given in the following result.

Theorem 2.14: For any nontrivial tree T, $\gamma_{ns}(T) \leq \gamma_{WB}(T) + \alpha_1(T) + 1$.

Proof: Let $A = \{e_1, e_2, e_3, \dots, e_n\} \subseteq E(T)$ be the edge set of T. Suppose $B = \{v_1, v_2, \dots, v_n\}$ be the set of vertices which are incident with the edges of A and if $|B| = p$. Then A itself is an edge covering number. Otherwise consider the minimum number of edges, $\{e_m\} \subseteq E(T) - A$ such that $A_1 = A \cup \{e_m\}$ forms a minimal edge covering set of T. Let $D = \{v_1, v_2, \ldots, v_n\} \subseteq V(T)$ is a dominating set T. If a vertex $v \epsilon D$ there exists a vertex $u \in V(T) - D$ such that $N(u) \cap D = \{v\}$ gives minimum nonsplit dominating set such that $|D| = \gamma_{ns}(T)$.

Suppose $\{b_1, b_2, b_3, \ldots, b_n\}$ be the set of vertices of $B(T)$ corresponding to the blocks $\{B_1, B_2, B_3, \ldots, B_n\}$ of T. Let $S = \{b_1, b_2, b_3, \ldots, b_m\}$ where $m < n$ is a minimal dominating set of $B(T)$ such that $V[B(T)] - S = N$, $\forall\, v_j \in N$, $\deg(v_i) \leq \deg(v_j)$, $\forall\, v_i \in S$. Then $|S| = \gamma_{WB}(T)$. Hence $|D| \leq |S| + |A_1| + 1$ gives $\gamma_{ns}(T) \leq \gamma_{WB}(T) + \alpha_1(T) + 1$.

Theorem 2.15: For any nontrivial tree T, $\gamma_{sns}(T) \leq \gamma_{WB}(T) + \gamma_t(T) + \left\lceil \frac{p}{2} \right\rceil + 2$.

Proof: Let $K = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V(T)$ such that $N[u_i] \cap N[u_j] = \emptyset$ where $1 \leq i \leq n$, $1 \leq j \leq n$. Suppose there exists a minimal set $K_1 = \{u_1, u_2, u_3, \ldots, u_k\} \in N(K)$ such that the subgraph $< K \cup K_1 >$ has no isolated vertex. Further, if $K \cup K_1$ covers all the vertices in T, then $K \cup K_1$ form a minimal total dominating set of T. Now suppose each block of T is an edge. Then $V(T) = \{v_1, v_2, v_3, \ldots, v_n\}$ and there exists a set $H = \{v_1, v_2, v_3, \ldots, v_i\}$, $1 \leq i \leq n$, $H \subseteq V(T)$ such that $v_j, v_k \epsilon V(T)$ and $v_j, v_k \in E(T)$. Hence $V(T) - \{H\} = v_j, v_k$ is complete. Clearly H is $\gamma_{sns} - set$.

Suppose $A = \{e_1, e_2, e_3, \ldots, e_i\} = E(T)$ be the edge set of T. In $B(T)$, let $S = \{u_1, u_2, u_3, \ldots, u_n\} = V[B(T)]$ be the set of vertices corresponding to the edges of A. Suppose $D \subseteq S$ be the set of vertices with $\deg(w) \geq 1$ for every $w \epsilon D$. Assume there exists $D' \subseteq D$ such that $\forall\, u_j \in D'$ $\deg(u_j) \leq \deg(u_k)$, $\forall\, u_k \in V[B(T)] - D'$. Clearly D' forms a weak block dominating set of T. Hence $|H| \leq |D'| + |\{K \cup K_1\}| + \left\lceil \frac{V(T)}{2} \right\rceil + 2$. Clearly $\gamma_{nsn}(T) \leq \gamma_{WB}(T) + \gamma_t(T) + \left\lceil \frac{p}{2} \right\rceil + 2$.

In the following theorem we establish the relation with double domination number of T.

Theorem2.16: For any tree T, $\gamma_{dd}(T) \leq \gamma_{WB}(T) + q$. Equality holds for $K_{1,n}$.

Proof: Let D be the minimal dominating set of T. If $F = \{u_1, u_2, u_3, \ldots, u_k\}$ be the set of all end vertices in T. Then $F \cup H = D^d$ where $H \subseteq V(T) - F$ forms a double dominating set of T, such that $|N(u) \cap D^d| \geq 2$, $\forall \in V(T) - D^d$.

Let $E_1 = \{e_1, e_2, e_3, \ldots, e_n\}$ be the set of edges in T and $E_1 \subseteq E(T)$. Then in $B(T)$, $D = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices corresponding to E. Let $\deg(e_i)$, $\forall\, e_i \in E_1$ and $\deg(e_j)$ $\forall$ $e_j \in E(T) - E_1$ such that $\deg(e_i) \leq \deg(e_j)$. Suppose $D_1 = \{v_1, v_2, \ldots, v_i\} \subseteq D$ and $N[v_k] = V[B(T)]$, $\forall\, v_k \in D_1$, $1 \leq k \leq i$. Then D_1 forms $\gamma_{WB} - set$. It follows that $|D^d| \leq |D_1| + |E(T)|$. Hence $\gamma_{dd}(T) \leq \gamma_{WB}(T) + q$.

To see the sharpness consider stars of order at least three.

3. REFERENCES

[1] R. B. Allan and R. Laskar, On domination and independent domination number of a graphs, Discrete Mathematics, Vol-23, 1978, 73-76.

[2] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs, Networks, Vol-10, 1980, 211-219.

[3] E. J. Cockayne, P. A. Dreyer. Jr, S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs,Discrete Maths, V0l-278, 2004, 11-22.

[4] E. J. Cockayne, B. L. Hartnell, S. T. Hedetniemi and R. Laskar, Perpect domination in graphs, J. Combin. Inform. System Sci, Vol – 18, 1993, 136 – 148.

[5] G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. C. Laskar and L. R. Markus,Restrained domination in graphs, Discrete Mathematics, Vol- 203, 1999, 61-69.

[6] F. Harary,Graph Theory, Adison – Wesley, Reading Mass, 1974.

[7] F. Harary and T. W. Haynes, Double domination in graphs, Ars combin, Vol-55, 2000, 201- 213.

[8] J. H. Hattingh and M. A. Hanning, On strong domination in graphs, J. Combin. Math. Combin. Comput, Vol – 26, 1998, 73 – 92.

[9] J. H. Hattingh and R. C. Laskar, On weak domination in graphs, Ars Combinatoria, Vol – 49, 1998.

[10] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of domination in Graphs,Marcel Dekkar, Inc., New York, 1998.

[11] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), Domination in graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.

[12] S. T. Hedetniemi and R. C. Laskar, Connected domination in graphs, Graph theory and combinatories cambridge, (Academic press), Landon, 1948, 209- 218.

[13] V. R. Kulli, Theory of domination in graphs, Viswa international publications, India, 2010.

[14] V. R. Kulli and M. H. Muddebihal, Lict Graph and Litact Graph of a Graphs, Journal of Analysis and computation, Vol – 2, 2006, 133 - 143.

[15] S. L. Mitchell and S. T. Hedetniemi, Edge domination in Trees, Congruent Number, Vol-19, 1977, 489 – 509.

[16] M. H. Muddebihal and Geetadevi Baburao, Weak line domination in graph theory, IJRAR, Vol – 6, 2019, 129 – 133.

[17] M. H. Muddebihal and Nawazoddin U. Patel, Strong Lict domination in graphs, IJRASET, Vol – 45, 2017, 596 – 602.

[18] M. H. Muddebihal and Priyanka H. Mandarvadkar, Global domination in Line graphs, IJRAR, Vol - 6, 2019, 648 – 652.

[19] M. H. Muddebihal, G. Srinivas, A. R. Sedamkar, Domination in squares graphs, Ultra scientist, Vol – 23(3), 2011, 795 – 800.

[20] M. H. Muddebihal and Suhas P. Gade, Lict double domination in graphs,Global Journal of pure and applied Mathematics, Vol-13, 2017, 3113-3120.

[21] M. H. Muddebihal and Sumangaladevi, Roman Block domination in graphs, International journal for Research in Science and Advanced Technologies, Vol – 6, 2013, 267 – 275.

[22] M. H. Muddebihal and Vedula Padmavathi, Connected block domination in graphs, Int, J. of Physical Science, Vol – 25, 2013, 453 - 458.

[23] Mustapha Chellali and Nader Jafari Rad, Weak Total domination in graphs, Utilitas Mathematics, Vol – 94, 2014, 221 – 236.

[24] D. Routenbach, Bounds on the weak domination number, Austral J. Combin, Vol – 18, 1998, 245 – 251.

[25] D. Routenbach, Bounds on the strong domination number, Discrete Math, Vol – 215, 2000, 201 – 212.

[26] E. Sampathkumar and L. Puspa Latha, Strong weak domination number and domination balance in graphs, Discreate Math, Vol – 161, 1996, 235 – 242.