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Separable Programming to a Multivariate Allocation Problem

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ABSTRACT
In this paper the multivariate allocation problem with upper limits on the available costs for various characters is considered. This problem is formulated as a separable programming problem and then solving it by separable programming approach. A numerical illustration is also given.

Keywords
Multivariate allocation problem, non-linear programming, separable programming

1. INTRODUCTION
The problem of sample allocation in multivariate stratified sampling has drawn the attention of the researchers for a long time starting apparently with Neyman (1934) [1]. Several criteria of allocation are put forward in the literature. For example, Ghosh (1958) [2] has considered an allocation based on the minimization of generalized variance. Yates (1960) [3] has suggested a procedure in which the cost is minimized for given precision for the estimate of each characteristic. Cochran (1963) [4] suggested the use of the average of individual optimum allocation for various characters. Kokan and Khan (1967) [5] gave an analytical solution to the above problem. Rao, T. J. (1993) [6] has reviewed many of the available criteria of allocation and also suggested new procedures.

We consider the multivariate allocation problem when the costs of enumerating the different characters have been fixed as upper limits. The objectives to be minimized are the variances of the estimates for various characteristics. The problem is formulated as a non-linear programming problem with linear objective function and several non- linear convex constraints, which is of separable nature. In this paper, an approximate solution to this problem is obtained by using separable programming technique.

2. STATEMENT OF THE PROBLEM
Consider a multivariate population partitioned into L strata. Suppose that p characteristics are measured on each unit of the population. We assume that the strata boundaries are fixed in advance. Let n_i be the number of units drawn without replacement from i^{th} stratum $(i=1,2,...,L)$. Let N_i be the size of the i^{th} stratum.For j^{th} character, an unbiased estimate of the population mean $\bar{Y}_j\ (j=1,2,...,p)$, denoted by $\bar{y}_{jst}$ has its sampling variance

$$V(\bar{y}_{jst})=\sum_{i=1}^{L}\left(\frac{1}{n_i}-\frac{1}{N_i}\right)W_i^2S_{ij}^2 \qquad (2.1)$$

Where $W_i=\frac{N_i}{N},\ S_{ij}^2=\frac{1}{N_i-1}\sum_{k=1}^{N_i}\left(y_{ijk}-\bar{Y}_{ij}\right)^2$

Substituting $a_{ij}=W_i^2S_{ij}^2$, we get

$$V(\bar{y}_{jst})=\sum_{i=1}^{L}\frac{a_{ij}}{n_i}-\sum_{i=1}^{L}\frac{a_{ij}}{N_i}$$

Let C_{ij} be the cost of enumerating the j^{th} character in the i^{th} stratum and let C_j be the amount received for j^{th} character. Then assuming linear cost function, the cost of the survey may be expressed $\sum_{i=1}^{L}C_{ij}n_i\le C_j$,

$$j=1,2,...,p \qquad (2.2)$$

Further, the survey is to be conducted in such a way that the variances for all the p characteristics are minimized.

$$\sum_{i=1}^{L}\frac{a_{ij}}{n_i}, \qquad (N_i{}^{'s}\text{ are given})$$

Where $a_{ij}=W_i^2S_{ij}^2$,

Using the transformation $\frac{1}{n_i}=x_i$, the allocation problem reduces to the following multi-objective convex programming problem

$$\left.\begin{array}{ll}\text{Minimize } V_j=\sum_{i=1}^{L}a_{ij}x_i, & \\ \text{Subject to } \sum_{i=1}^{L}\frac{C_{ij}}{x_i}\le C_j, & j=1,2,...,p \\ \frac{1}{N_i}\le x_i\le 1, & i=1,2,...,L\end{array}\right\} \qquad (2.3)$$

As both the objective function and the constraints are separable, it is possible to write the problem (2.3) in the form of a separable programming problem as follows:

$$\left.\begin{array}{ll}\text{Minimize } \sum_{i=1}^{L} f_{ij}(x_i), & \\ \text{Subject to } \sum_{i=1}^{L} g_{ij}(x_i) \le C_j, & j=1,2,...,p \\ \frac{1}{N_i} \le x_i \le 1, & i=1,2,...,L\end{array}\right\} \quad (2.4)$$

Where $f_{ij}(x_i) = a_{ij}x_i$,

$$g_{ij}(x_i) = \frac{C_{ij}}{x_i}, \qquad i=1,2,...,L$$

3. SOLUTION USING SEPARABLE PROGRAMMING

The non-linear functions $g_{ij}(x_i)$ can be approximated by piecewise linear functions. Let the feasible range of the variables be x_i, given by the interval $[\alpha_i, \beta_i]$ and choose a set of m_i grid points α_{ir} $(r=1,2,,...,m_i)$ such that $\alpha_i = \alpha_{i1} < \alpha_{i2} < ... < \alpha_{im_i} = \beta_i$.

Every point x_i in the grid $[\alpha_{ir}, \alpha_{ir+1}]$ can be expressed as

$$x_i = \lambda_{ir}\,\alpha_{ir} + \lambda_{i,r+1}\,\alpha_{i,r+1} \qquad (3.1)$$

Where $\lambda_{ir} + \lambda_{i,r+1} = 1$

and $\lambda_{ir} \ge 0, \lambda_{i,r+1} \ge 0$

A linear approximation for a function $g_{ij}(x_i)$ in the grid interval is $[\alpha_{ir}, \alpha_{ir+1}]$ is $\hat{g}_{ij}(x_i) = \lambda_{ir} g_{ij}(\alpha_{ir}) + \lambda_{i,r+1}\, g_{ij}(\alpha_{i,r+1})$. In general for $x_i \varepsilon [\alpha_i, \beta_i]$, the piecewise linear approximation $\hat{g}_{ij}(x_i)$ can be written as

$$\hat{g}_{ij}(x_i) = \sum_{r=1}^{m_i} \lambda_{ir} g_{ij}(\alpha_{ir}) \qquad (3.2)$$

With $x_i = \sum_{r=1}^{m_i} \lambda_{ir}\alpha_{ir}$, $\sum_{r=1}^{m_i} \lambda_{ir} = 1;\ \lambda_{ir} \ge 0,$

For $r = 1,2,...,m_i$

Provided for each, i at the most two adjacent λ_{ir} are positive.

An approximating linear program to the non-linear separable program (2.4) is thus obtained as

$$\left.\begin{array}{ll}\text{Minimize } Z = \sum_{i=1}^{L}\sum_{r=1}^{m_i} \lambda_{ir} g_{ij}(\alpha_{ir}) & \\ \text{Subject to } \sum_{i=1}^{L}\sum_{r=1}^{m_i} \lambda_{ir} g_{ij}(\alpha_{ir}) \le C_j, & j=1,2,...,p \\ \sum_{r=1}^{m_i} \lambda_{ir} = 1, \lambda_{ir} \ge 0, & r=1,2,...,m_i\end{array}\right\} \quad (3.3)$$

Problem (3.3) is a linear programming problem and can be solved by simplex method using restricted basis entry rule for separable functions Hadley (1964)[7].Theoptimal values of $\lambda_{ir}^{*}(r=1,2,...,m_i\ \&\ i=1,2,...,L)$ obtained by solving problem (3.3) yield an approximate optimal solution $\hat{X}$ to the original problem (2.4) as

$$\hat{x}_i = \sum_{r=1}^{n_i} \lambda_{ir}^{*}\alpha_{ir}, \qquad i=1,2,...,L$$

Example: Consider an allocation problem with one character $(p=1)$ & two strata $(L=2)$, with following information:

i	W_i	S_{i1}	S_{i2}	C_{i1}	C_{i2}	N_i
1	0.3	2	3	0.6	1.5	18
2	0.45	4	1	0.8	2	27

The variance coefficients matrix is given by:

$$(a_{ij}) = \begin{pmatrix} 0.36 & 0.81 \\ 3.24 & 0.2025 \end{pmatrix}$$

Let us fix the budget at 80 and 100 units.

Using the above information the problem in (2.3) will be as follows:

Thus,

$$\begin{aligned}&\text{Min. } V = 1.17x_1 + 3.4425x_2\\ &\text{s.t. } \frac{0.6}{x_1} + \frac{0.8}{x_2} \le 80\\ &\quad \frac{1.5}{x_1} + \frac{2}{x_2} \le 100\\ &\quad 0.055 \le x_1 \le 1\\ &\quad 0.037 \le x_2 \le 1\end{aligned}$$

$$\alpha_{11} = 0.055,\ \alpha_{12} = 0.2912,\ \alpha_{13} = 0.5275,\ \alpha_{14} = 0.7367,\ \alpha_{15} = 1$$
$$\alpha_{21} = 0.055,\ \alpha_{22} = 0.2912,\ \alpha_{23} = 0.5275,\ \alpha_{24} = 0.7367,\ \alpha_{25} = 1$$

The piecewise linear approximation to the function $f_1(x_1) = 1.17x_1,\ f_2(x_2) = 3.4425x_2$ $g_{11}(x_1) = \frac{0.6}{x_1},\ g_{12}(x_2) = \frac{0.8}{x_2},\ g_{21}(x_1) = \frac{1.5}{x_1},\ g_{22}(x_2) = \frac{2}{x_2}$ are

$$\hat{f}_1(x_1) = 0.064\lambda_{11} + 0.341\lambda_{12} + 0.617\lambda_{13} + 0.862\lambda_{14} + 1.17\lambda_{15}$$
$$\hat{f}_2(x_2) = 0.189\lambda_{21} + 1.003\lambda_{22} + 1.816\lambda_{23} + 2.536\lambda_{24} + 3.443\lambda_{25}$$
$$\hat{g}_{11}(x_1) = 10.909\lambda_{11} + 2.060\lambda_{12} + 1.137\lambda_{13} + 0.814\lambda_{14} + 0.6\lambda_{15}$$
$$\hat{g}_{12}(x_2) = 14.545\lambda_{21} + 2.747\lambda_{22} + 1.517\lambda_{23} + 1.086\lambda_{24} + 0.8\lambda_{25}$$
$$\hat{g}_{21}(x_1) = 27.273\lambda_{11} + 5.151\lambda_{12} + 2.844\lambda_{13} + 1.964\lambda_{14} + 1.5\lambda_{15}$$
$$\hat{g}_{22}(x_2) = 36.364\lambda_{21} + 6.868\lambda_{22} + 3.791\lambda_{23} + 2.619\lambda_{24} + 2.0\lambda_{25}$$

Thus the approximated separable programming problem becomes

$$\begin{aligned}&Minimize\ \ \hat{f}_1(x_1) + \hat{f}_2(x_2)\\ &\text{s.t.} \quad \hat{g}_{11}(x_1) + \hat{g}_{12}(x_2) \le 80\\ &\quad \hat{g}_{21}(x_1) + \hat{g}_{22}(x_2) \le 100\\ &\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} = 1\\ &\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} = 1\end{aligned}$$

Solving the above linear programming problem by simplex method using restricted basis entry rule. The optimal solution is given by

$$\lambda_{11}=1,\ \lambda_{12}=0,\ \lambda_{13}=0,\ \lambda_{14}=0,\ \lambda_{15}=0$$

$$\lambda_{21}=1,\ \lambda_{22}=0,\ \lambda_{23}=0,\ \lambda_{24}=0,\ \lambda_{25}=0$$

Thus, $$x_1=\sum_{r=1}^{5}\lambda_{1r}\alpha_{1r}=0.055$$

$$x_2=\sum_{r=1}^{5}\lambda_{2r}\alpha_{2r}=0.055$$

With $\text{Min. } Z=0.2537$

4. CONCLUSION

In this particular example the optimal solution for the varying intervals(3,5 and 9) is same i.e., MinimumZ=0.2537with $x_1=0.055\ \ and\ \ x_2=0.055$

5. FURTHER STUDY

This can be checked for other examples whether the no. of intervals make any difference to the optimality of the solution or not and if us then how many no. of intervals.

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